Functions

Part Two

Outline for Today

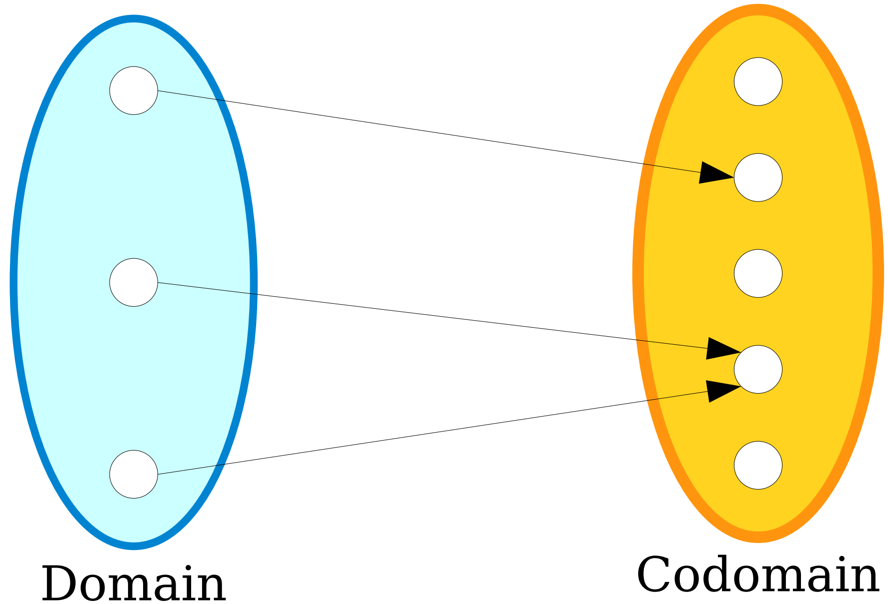
- ***Recap from Last Time***
  - Where are we, again?
- ***Connecting Function Types***
  - Relating the topics from last time.
- ***Function Composition***
  - Sequencing functions together.

Recap from Last Time

Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, $f(x)$ belongs to the codomain.
- We write $\boldsymbol{f : A \to B}$ to indicate that *f* is a function whose domain is *A* and whose codomain is *B*.

The function must be defined for each element of its domain.



The output of the function must always be in the codomain, but not all elements of the codomain need to be produceable.

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** if the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- For example, $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = -x$ is an involution.

Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) if different inputs always map to different outputs.
  - A function with this property is called an ***injection***.
- Formally, $f : A \to B$ is an injection if this FOL statement is true:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different"*)

- Equivalently:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same"*)

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) if each element of the codomain is "covered" by at least one element of the domain.
  - A function with this property is called a ***surjection***.
- Formally, $f : A \to B$ is a surjection if this FOL statement is true:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output, there's at least one possible input that produces it"*)

Proving vs. Assuming

- In the context of a proof, you will need to assume some statements and prove others.
  - We ***assumed*** all birds can fly.
  - We ***proved*** all herons can fly.
- Statements behave differently based on whether you’re assuming or proving them.

$$\underbrace{(\forall b.\ (Bird(b) \to CanFly(b)))}_{\text{We never introduce a variable } b.} \to \underbrace{(\forall h.\ (Heron(h) \to CanFly(h)))}_{\text{We introduce a variable } h \text{ almost immediately.}}$$

	To ***prove*** that this is true…	If you ***assume*** this is true…
$\forall x.\ A$	Have the reader pick an arbitrary x. We then prove A is true for that choice of x.	Initially, ***do nothing***. Once you find a z through other means, you can state it has property A.
$\exists x.\ A$	Find an x where A is true. Then prove that A is true for that specific choice of x.	Introduce a variable x into your proof that has property A.
$A \to B$	Assume A is true, then prove B is true.	Initially, ***do nothing***. Once you know A is true, you can conclude B is also true.
$A \land B$	Prove A. Then prove B.	Assume A. Then assume B.
$A \lor B$	Either prove $\neg A \to B$ or prove $\neg B \to A$. *(Why does this work?)*	Consider two cases. Case 1: A is true. Case 2: B is true.
$A \leftrightarrow B$	Prove $A \to B$ and $B \to A$.	Assume $A \to B$ and $B \to A$.
$\neg A$	Simplify the negation, then consult this table on the result.	Simplify the negation, then consult this table on the result.

New Stuff!

Connecting Function Types

Types of Functions

- Last time, we saw three special types of functions:
  - ***involutions***, functions that undo themselves;
  - ***injections***, functions where different inputs go to different outputs; and
  - ***surjections***, functions that cover their whole codomain.
- ***Question:*** How do these three classes of functions relate to one another?

Theorem: For any function $f : A \to A$, if f is an involution, then f is surjective.

$$\underbrace{(\forall x \in A.\ f(f(x)) = x)}_{\text{f is an involution.}} \quad \to \quad \underbrace{(\forall b \in A.\ \exists a \in A.\ f(a) = b)}_{\text{f is surjective.}}$$

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Since we're ***assuming*** this, we aren't going to pick a specific choice of *x* right now. Instead, we're going to keep an eye out for something to apply this fact to.

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Prove this.

We've said that we need to ***prove*** this statement. How do we do that?

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There's a universal quantifier up front. Since we're ***proving*** this, we'll pick an arbitrary $b \in A$.

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Now, we hit an existential quantifier. Since we're ***proving*** this, we need to find a choice of $a \in A$ where this is true.

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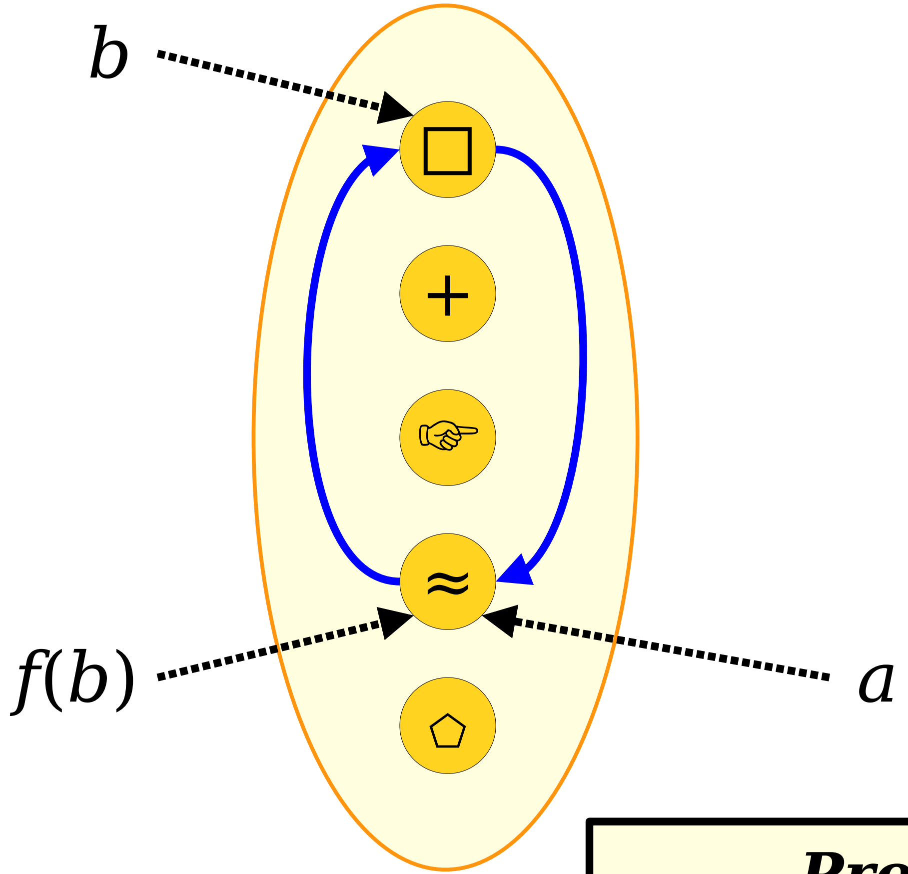

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Come up with a proof outline for this theorem. What variables will we introduce, what do we assume about them, and what is our Want to Show?

Respond at pollev.com/zhenglian740

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We need to ***prove*** this part.
What does that mean?

Prove
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We now need to ***prove*** this implication. But we know how to do that! We assume the antecedent and prove the consequent.

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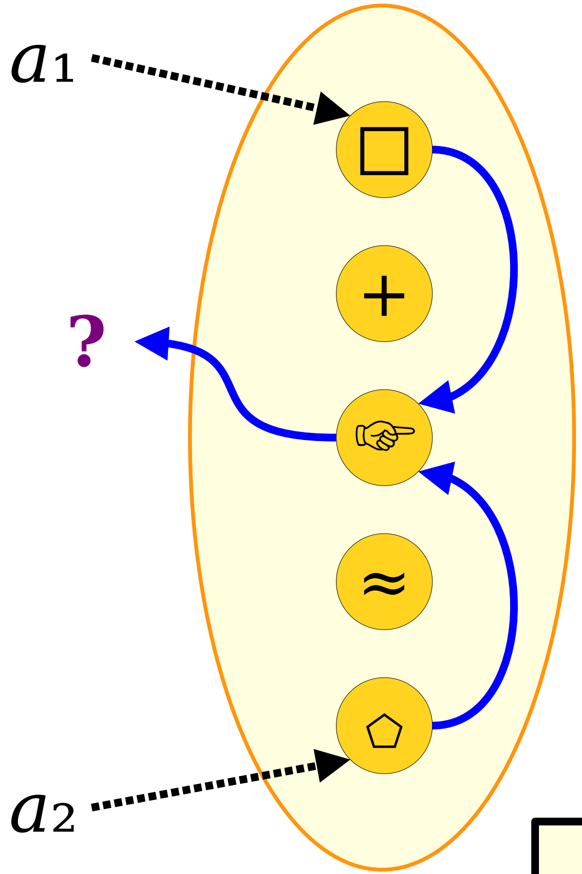

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We've reached a contradiction, so our assumption was wrong.

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Proof: Consider any function $f : A \to A$ that's an involution. We will prove that f is injective. To do so, choose any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We need to show that $f(a_1) \neq f(a_2)$.

We'll proceed by contradiction. Suppose that $f(a_1) = f(a_2)$. This means $f(f(a_1)) = f(f(a_2))$, which in turn tells us $a_1 = a_2$ because f is an involution. But that's impossible, since $a_1 \neq a_2$.

We've reached a contradiction, so our assumption was wrong. Therefore, we see that $f(a_1) \neq f(a_2)$, as required.

Proof Outline

1. Assume f is an involution.
2. Pick arbitrary $a_1, a_2 \in A$ such that $a_1 \neq a_2$.
3. Prove $f(a_1) \neq f(a_2)$.

Theorem: For any function $f : A \to A$, if f is an involution, then f is injective.

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Proof Outline

1. Assume f is an involution.
2. Pick arbitrary $a_1, a_2 \in A$ such that $a_1 \neq a_2$.
3. Prove $f(a_1) \neq f(a_2)$.

Let’s take a quick break!

Time-Out for Announcements!

Midterm Exam Logistics

- Our midterm exam will be on Friday, July 26th from 5:00 – 8:00 PM in Hewlett 201 (our normal lecture room).
- You're responsible for lectures up to the end of week 3 and topics from PS1 – PS3. Later lectures and problem sets won't be tested here. Exam problems may build on the written or coding components from the problem sets.

Midterm Exam Logistics

- The exam is open-book, open-note, and closed-other-humans/AI.
- You are free to make use of all course materials on the course website and on Canvas, including lecture slides and lecture videos. You are also permitted to search online for conceptual information (for example, by visiting Wikipedia).
- You are not permitted to communicate with other humans about the exam or to solicit help from others. For example, you must not communicate with other students in the course, you must not ask questions on sites like Chegg or Stack Overflow, and you must not receive assistance from any AI chatbots.

Midterm Exam

- ***We want you to do well on this exam.*** We're not trying to weed out weak students. We're not trying to enforce a curve where there isn't one. We want you to show what you've learned up to this point so that you get a sense for where you stand and where you can improve.
- The purpose of this midterm is to give you a chance to show what you've learned in the past few weeks. It is not designed to assess your “mathematical potential” or “innate mathematical ability.”

OAE Accommodations

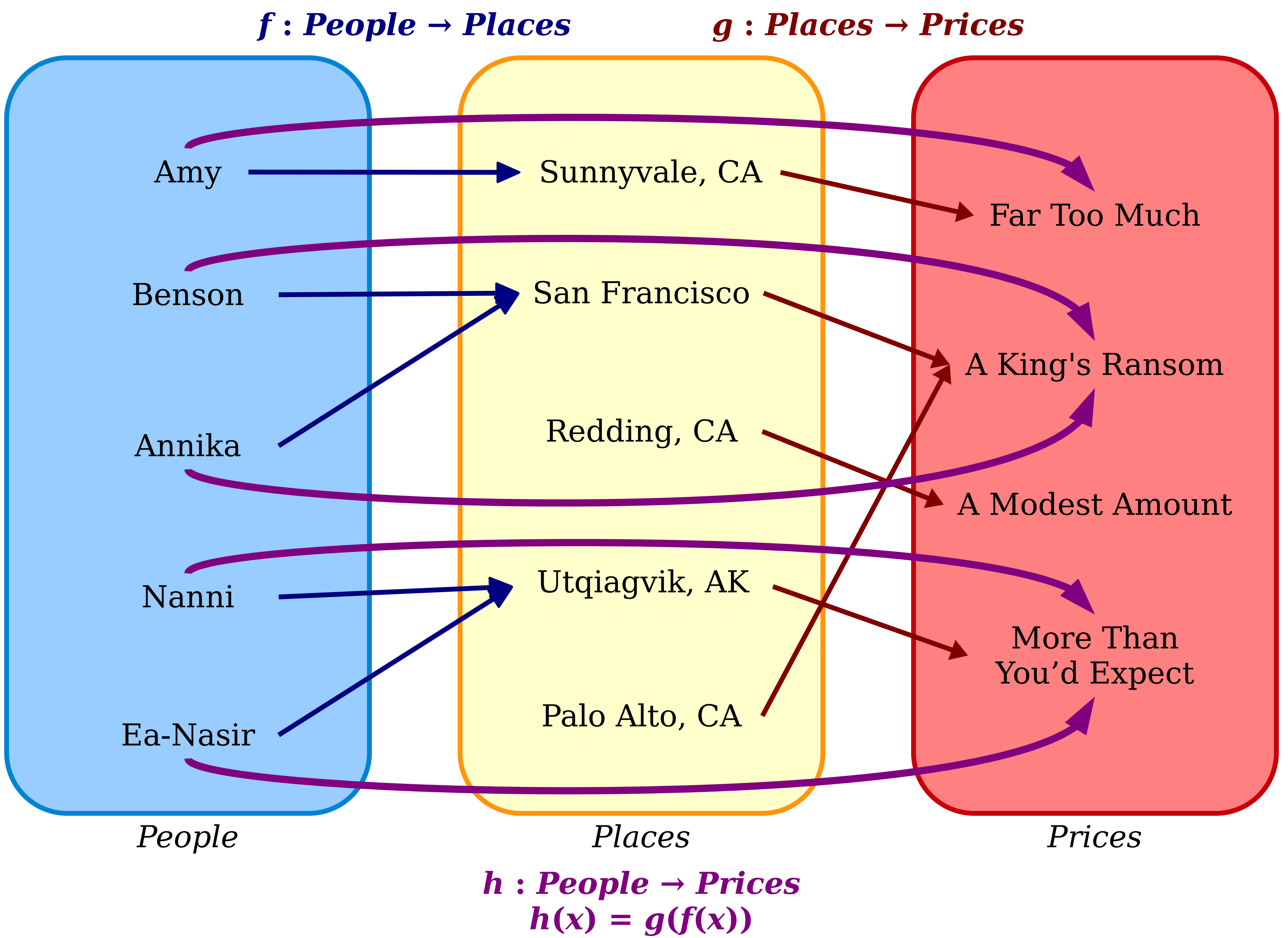
- We are currently in the process of reserving rooms for the midterm exam.
- If you have an OAE letter, please send it to cs103-sum2324-staff@lists.stanford.edu ASAP.
- We’ll be in touch in the upcoming week regarding room logistics.

Extra Practice Problems

- Up on the course website, you'll find some Extra Practice Problems on the topics covered by the upcoming midterm.
- Many of these are old midterm questions. Some are just really fun problems we thought you might enjoy working through.
- Take the time to work through some of these problems. We also released midterms from some of the previous quarters.

Back to CS103!

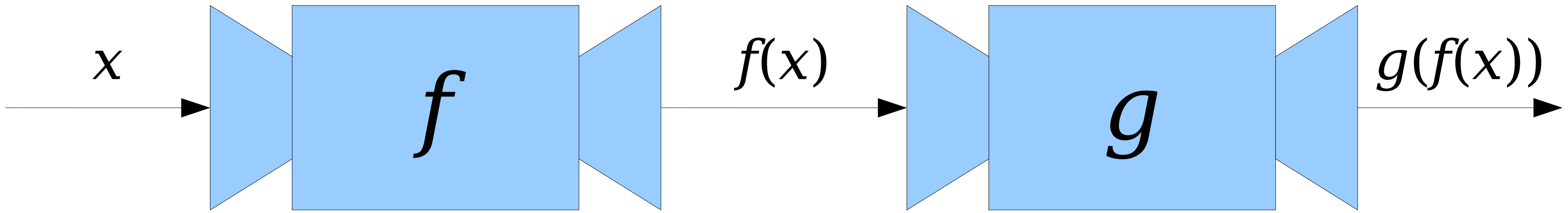
Function Composition



f : People → Places
g : Places → Prices
Amy
Benson
Annika
Nanni
Ea-Nasir
Sunnyvale, CA
San Francisco
Redding, CA
Utqiagvik, AK
Palo Alto, CA
Far Too Much
A King's Ransom
A Modest Amount
More Than You’d Expect
People
Places
Prices
h : People → Prices
h(x) = g(f(x))

Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- Notice that the codomain of f is the domain of g. This means that we can use outputs from f as inputs to g.



Function Composition

- Suppose that we have two functions $f : A \to B$ and $g : B \to C$.
- The ***composition of f and g***, denoted $\boldsymbol{g \circ f}$, is a function where
  - $g \circ f : A \to C$, and
  - $(g \circ f)(x) = g(f(x))$.

The name of the function is $g \circ f$. When we apply it to an input x, we write $(g \circ f)(x)$. I don't know why, but that's what we do.

- A few things to notice:
  - The domain of $g \circ f$ is the domain of f. Its codomain is the codomain of g.
  - Even though the composition is written $g \circ f$, when evaluating $(g \circ f)(x)$, the function f is evaluated first.

Properties of Composition

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

Organizing Our Thoughts

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming

$f : A \to B$ is an injection.

$\forall x \in A.\ \forall y \in A.\ (x \neq y \to$
$f(x) \neq f(y)$
$)$

$g : B \to C$ is an injection.

$\forall x \in B.\ \forall y \in B.\ (x \neq y \to$
$g(x) \neq g(y)$
$)$

We're ***assuming*** these universally-quantified statements, so we won't introduce any variables for what's here.

What We Need to Prove

$g \circ f$ is an injection.

$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
$(g \circ f)(a_1) \neq (g \circ f)(a_2)$
$)$

We need to ***prove*** this universally-quantified statement. So let's introduce arbitrarily-chosen values.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***

$f : A \to B$ is an injection.

$$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$$

$g : B \to C$ is an injection.

$$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$$

$a_1 \in A$ is arbitrarily-chosen.

$a_2 \in A$ is arbitrarily-chosen.

$g \circ f$ is an injection.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$$

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$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to$
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$)$

Now we're looking at an implication. Let's ***assume*** the antecedent and ***prove*** the consequent.

Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is an injection.

What We're Assuming	***What We Need to Prove***
$f : A \to B$ is an injection.	$g \circ f$ is an injection.
$\forall x \in A.\ \forall y \in A.\ (x \neq y \to f(x) \neq f(y))$	$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to (g \circ f)(a_1) \neq (g \circ f)(a_2))$
$g : B \to C$ is an injection.	Let's write this out separately and simplify things a bit.
$\forall x \in B.\ \forall y \in B.\ (x \neq y \to g(x) \neq g(y))$	
$a_1 \in A$ is arbitrarily-chosen.	
$a_2 \in A$ is arbitrarily-chosen.	
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$f : A \to B$ is an injection.	$g \circ f$ is an injection.
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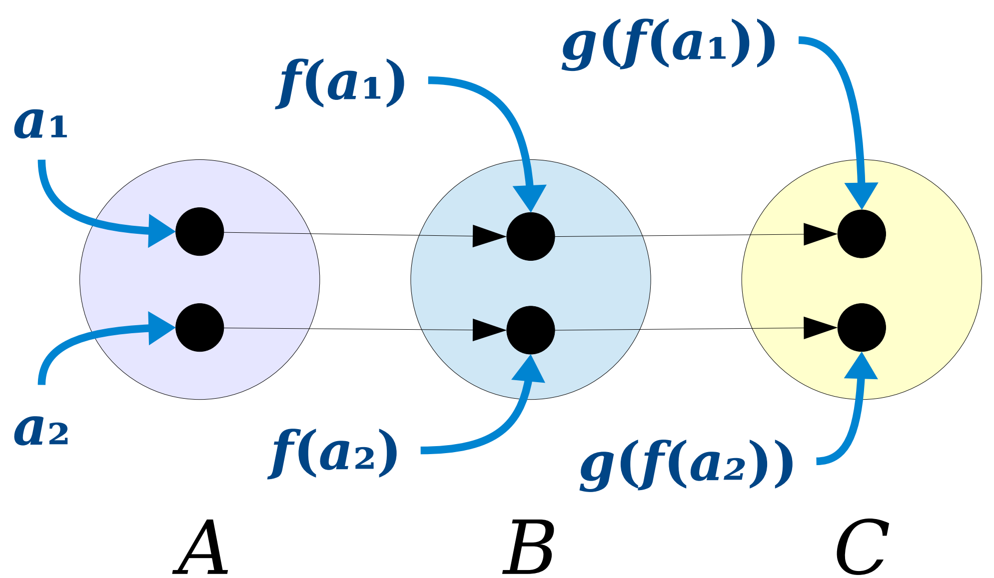
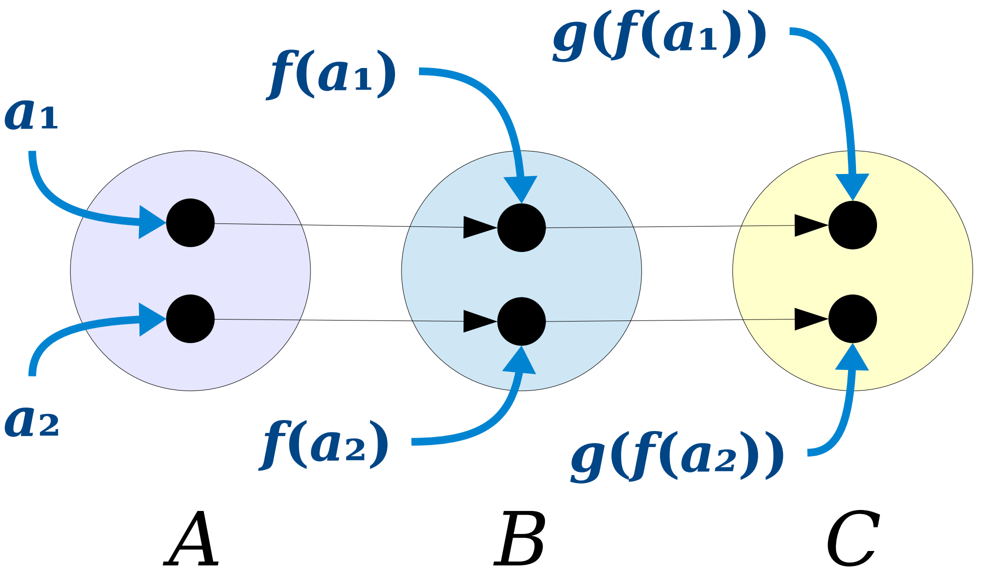


Theorem: If $f : A \to B$ is an injection and $g : B \to C$ is an injection, then the function $g \circ f : A \to C$ is also an injection.



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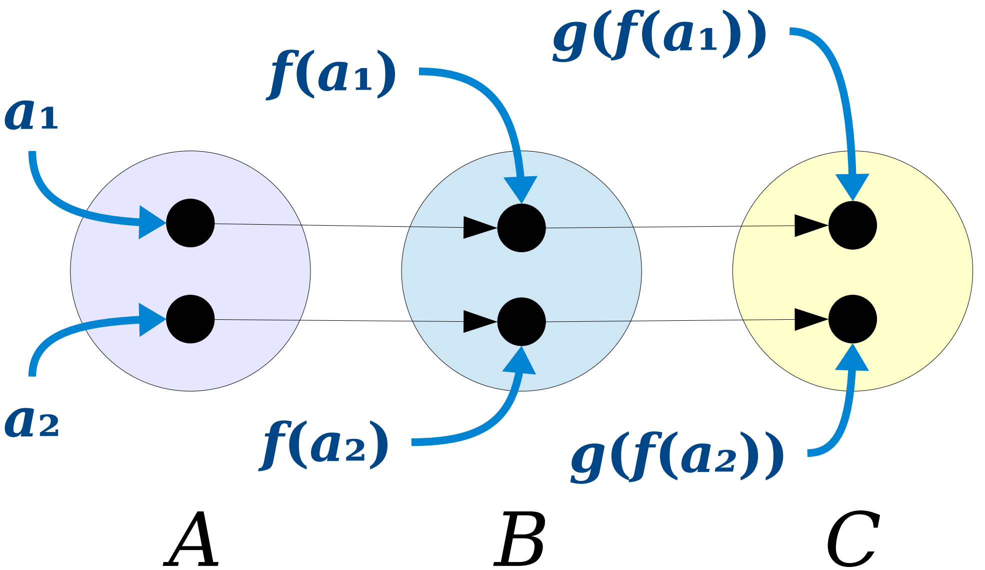
Proof:



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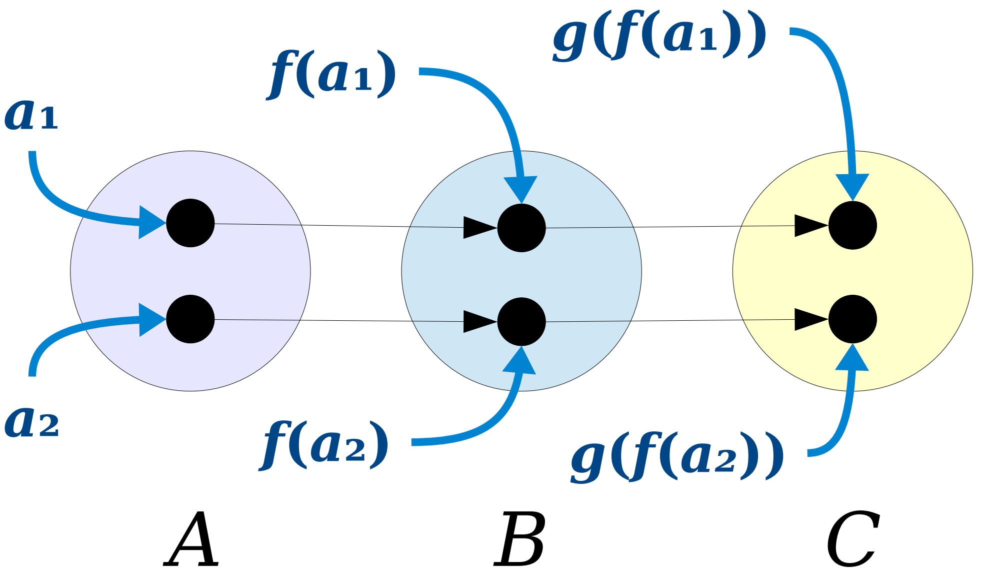
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections.



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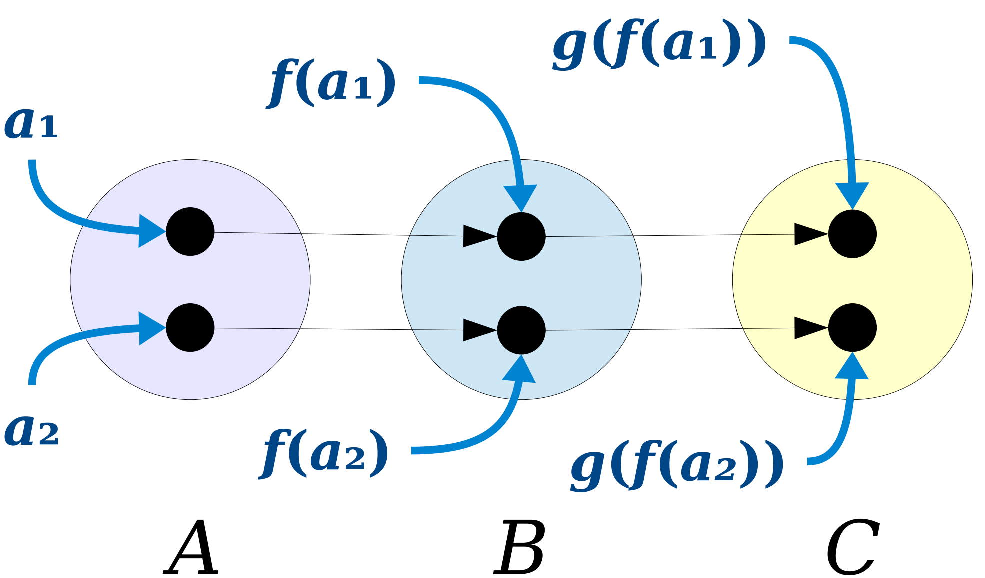

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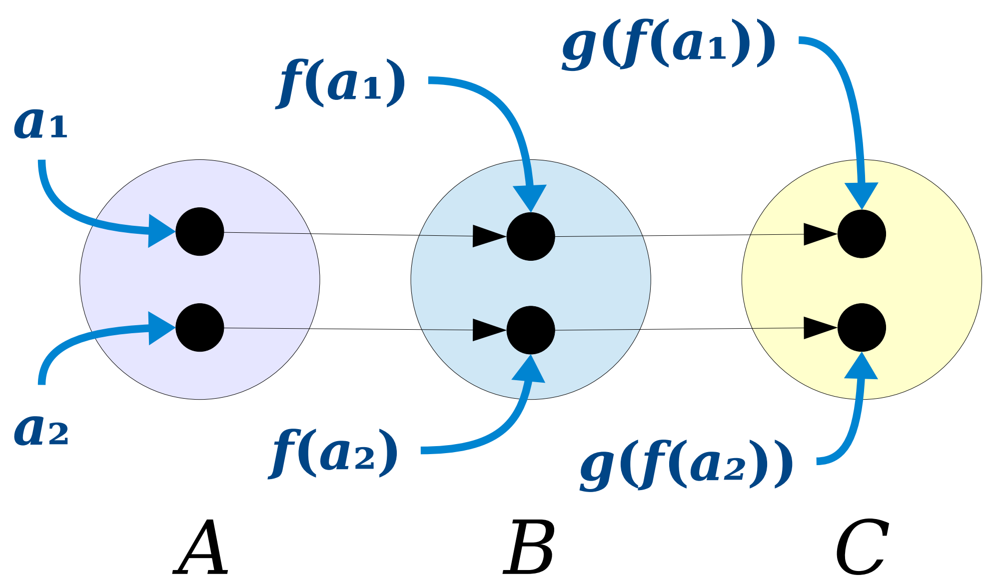

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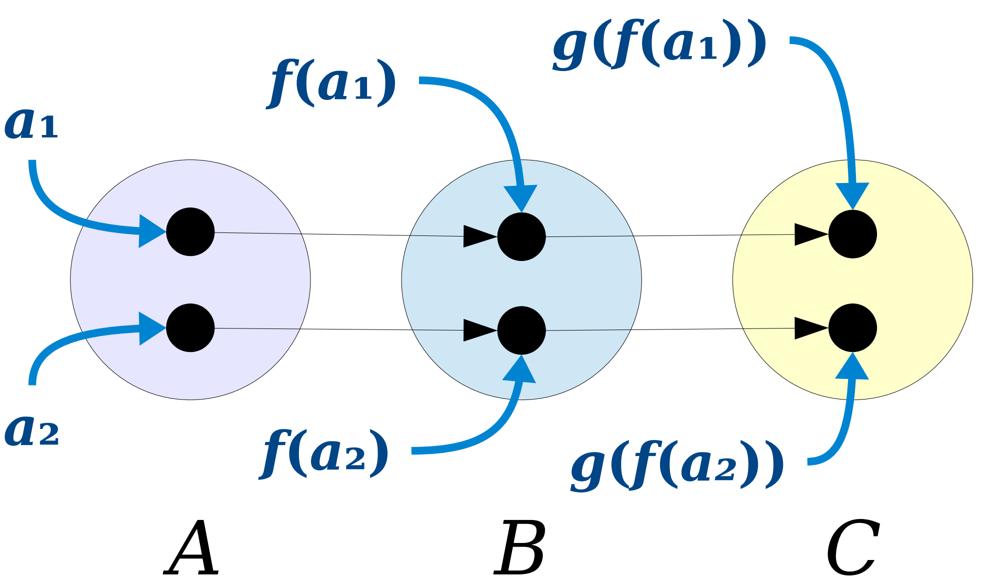

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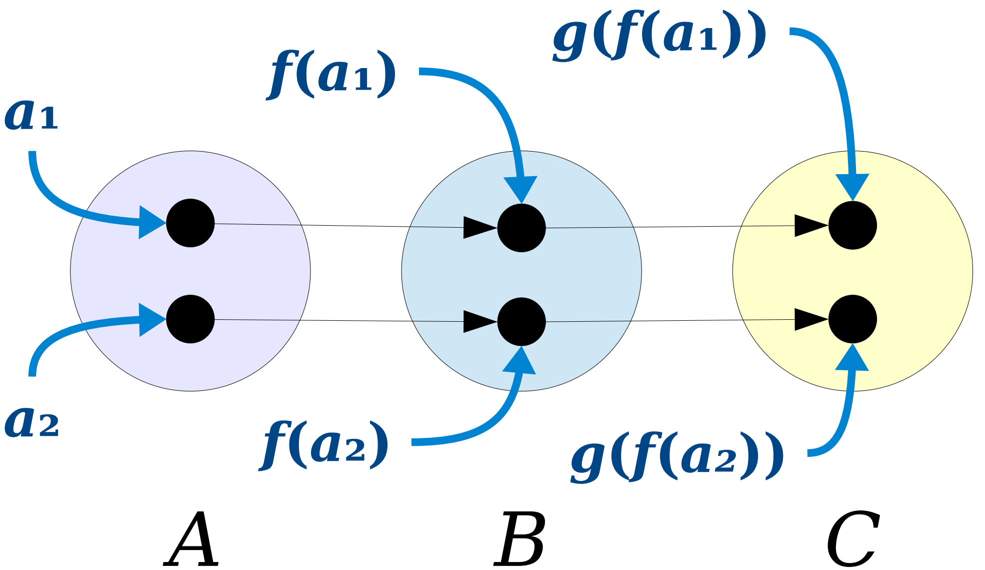

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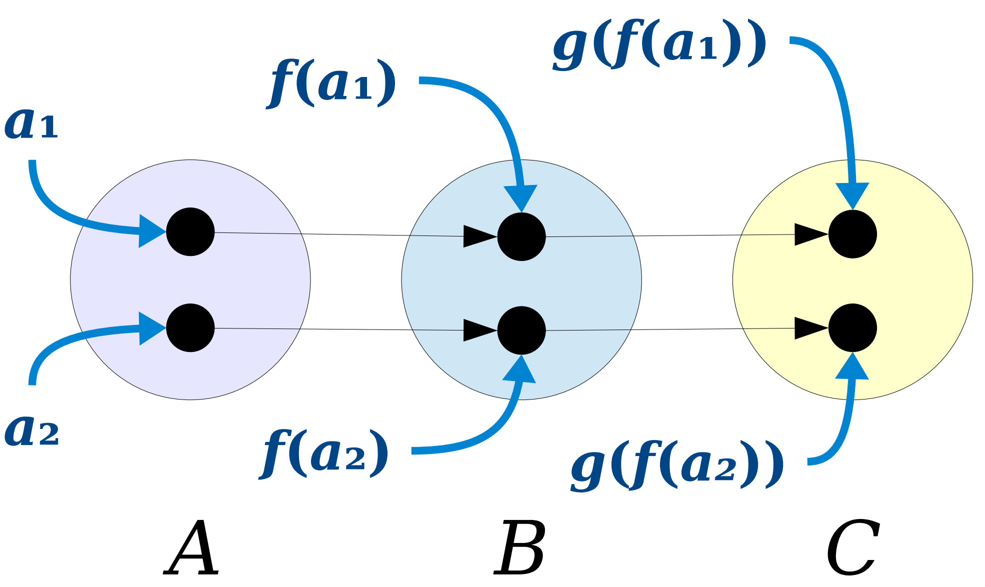
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$.



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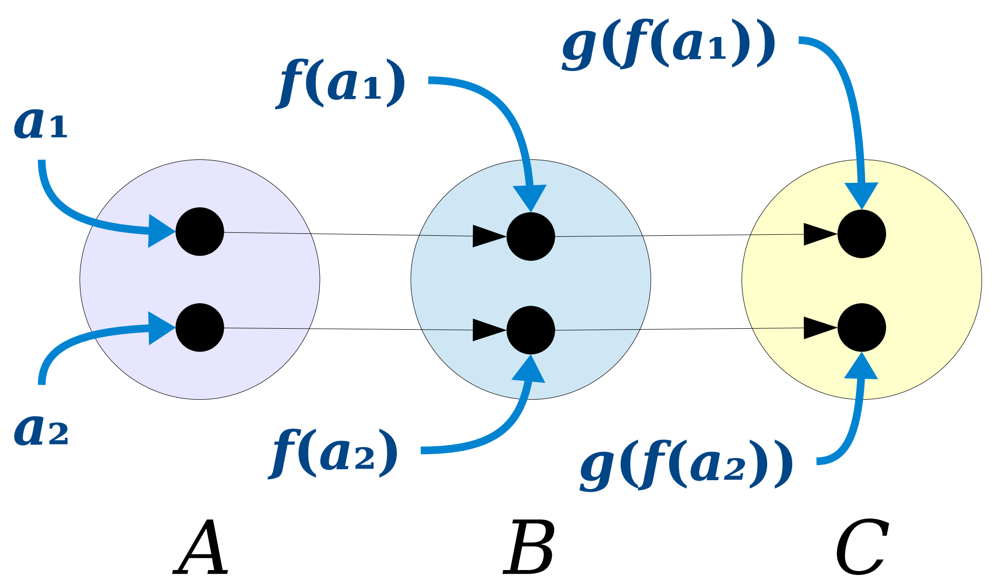
Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required.



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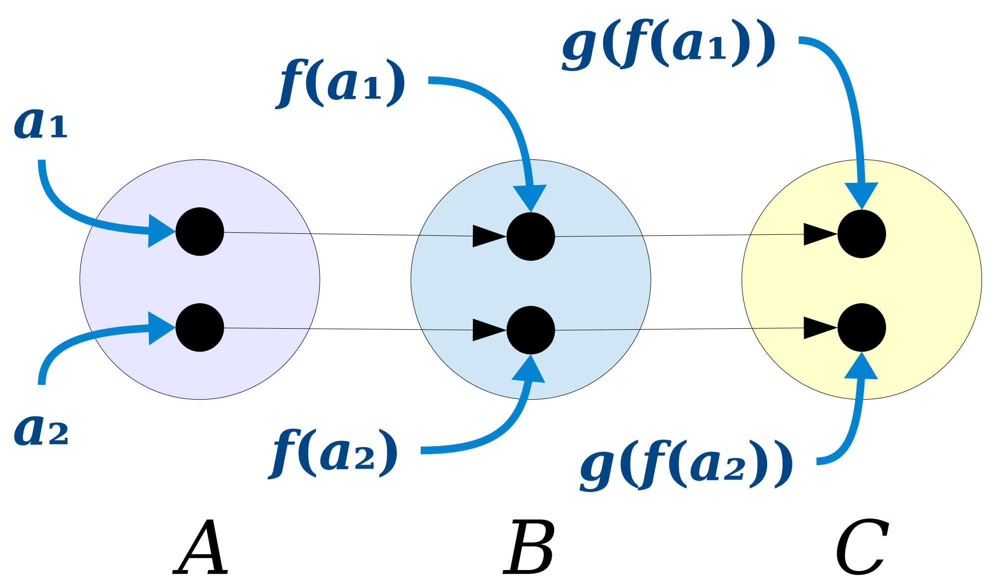

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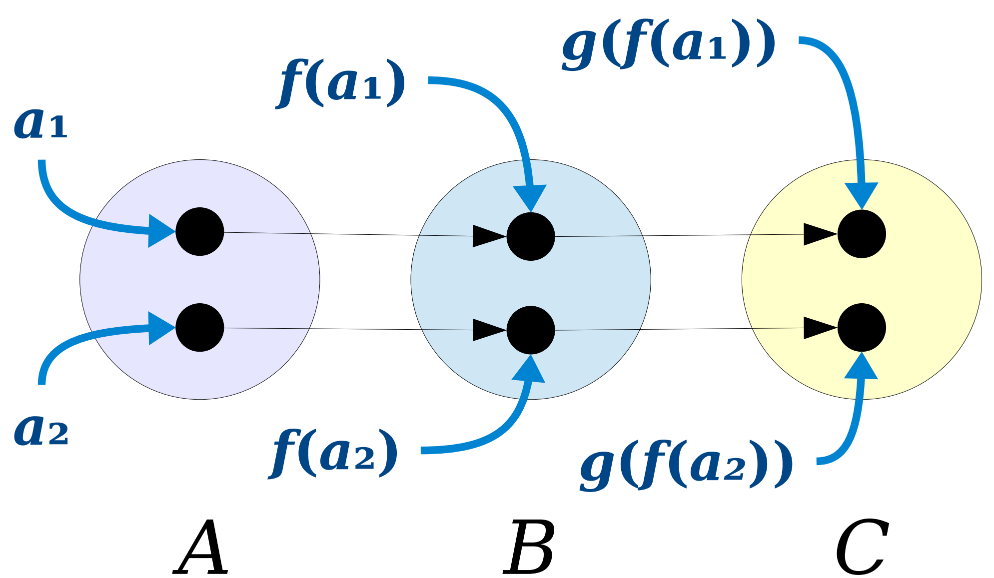
Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary injections. We will prove that the function $g \circ f : A \to C$ is also injective. To do so, consider any $a_1, a_2 \in A$ where $a_1 \neq a_2$. We will prove that $(g \circ f)(a_1) \neq (g \circ f)(a_2)$. Equivalently, we need to show that $g(f(a_1)) \neq g(f(a_2))$.

Since f is injective and $a_1 \neq a_2$, we see that $f(a_1) \neq f(a_2)$. Then, since g is injective and $f(a_1) \neq f(a_2)$, we see that $g(f(a_1)) \neq g(f(a_2))$, as required. ■

Great exercise: Repeat this proof using the other definition of injectivity.



Theorem: If $f : A \to B$ is a surjection and $g : B \to C$ is a surjection, then the function $g \circ f : A \to C$ is a surjection.

Theorem: If $f : A \to B$ is surjective and $g : B \to C$ is surjective, then $g \circ f : A \to C$ is also surjective.

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How should we complete this sentence?

Respond at pollev.com/zhenglian740

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will

What does it mean for $g \circ f : A \to C$ to be surjective?

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Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will

What does it mean for $g \circ f : A \to C$ to be surjective?

$$\forall c \in C.\ \exists a \in A.\ (g \circ f)(a) = c$$

Therefore, we'll choose an arbitrary $c \in C$ and prove that there is some $a \in A$ such that $(g \circ f)(a) = c$.

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Proof: Let $f : A \to B$ and $g : B \to C$ be arbitrary surjections. We will prove that the function $g \circ f : A \to C$ is also surjective. To do so, we will prove that for any $c \in C$, there is some $a \in A$ such that $(g \circ f)(a) = c$.

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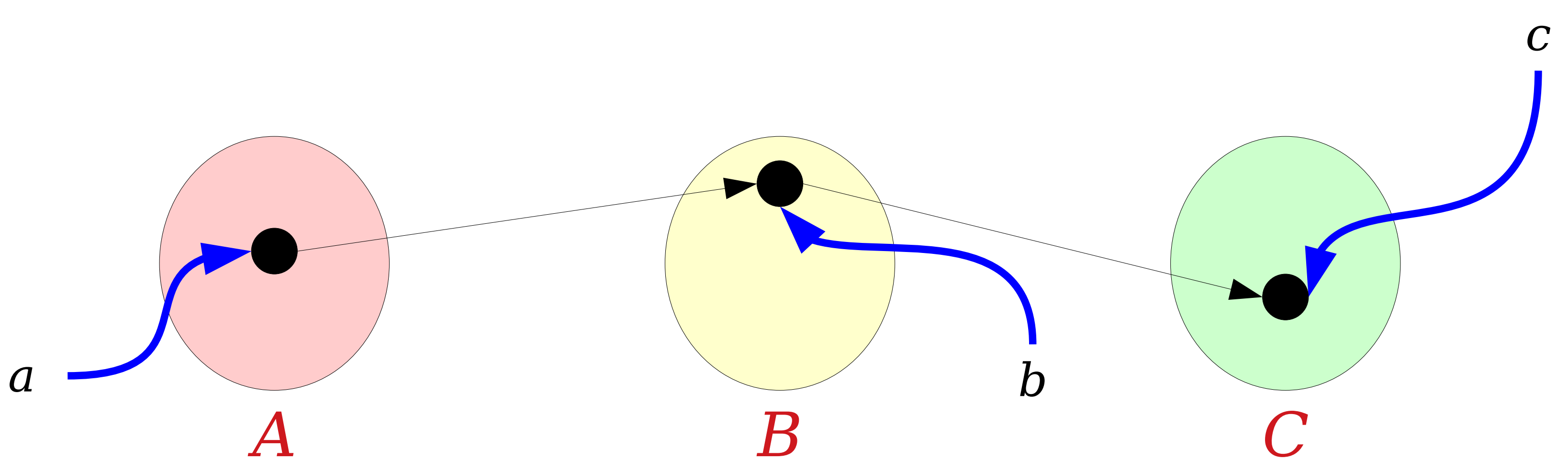

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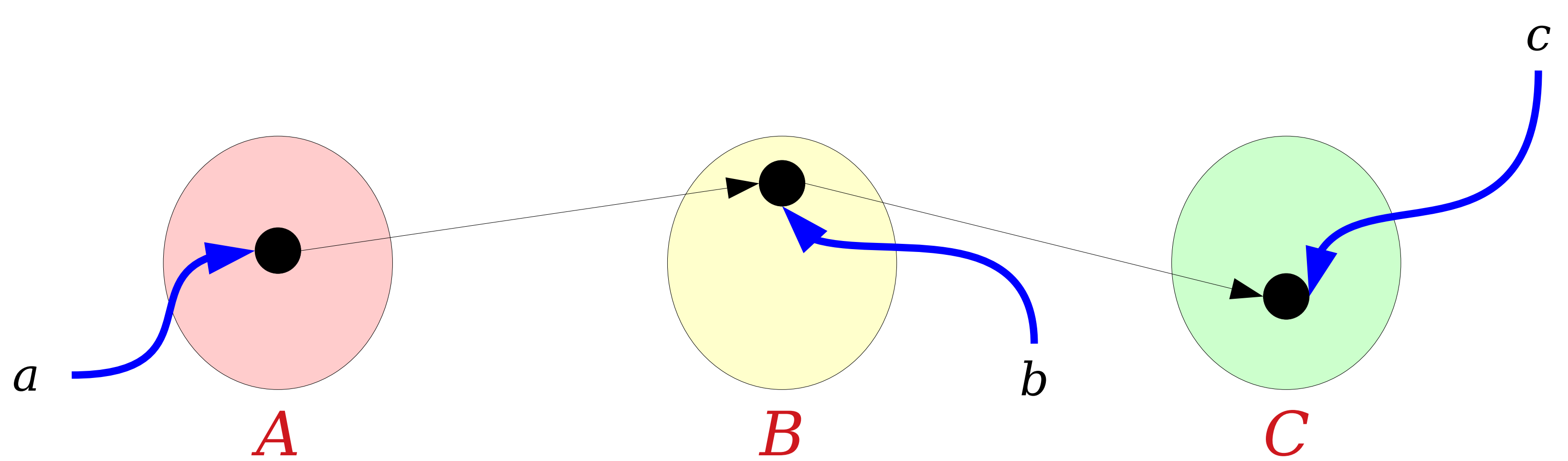
Consider any $c \in C$.



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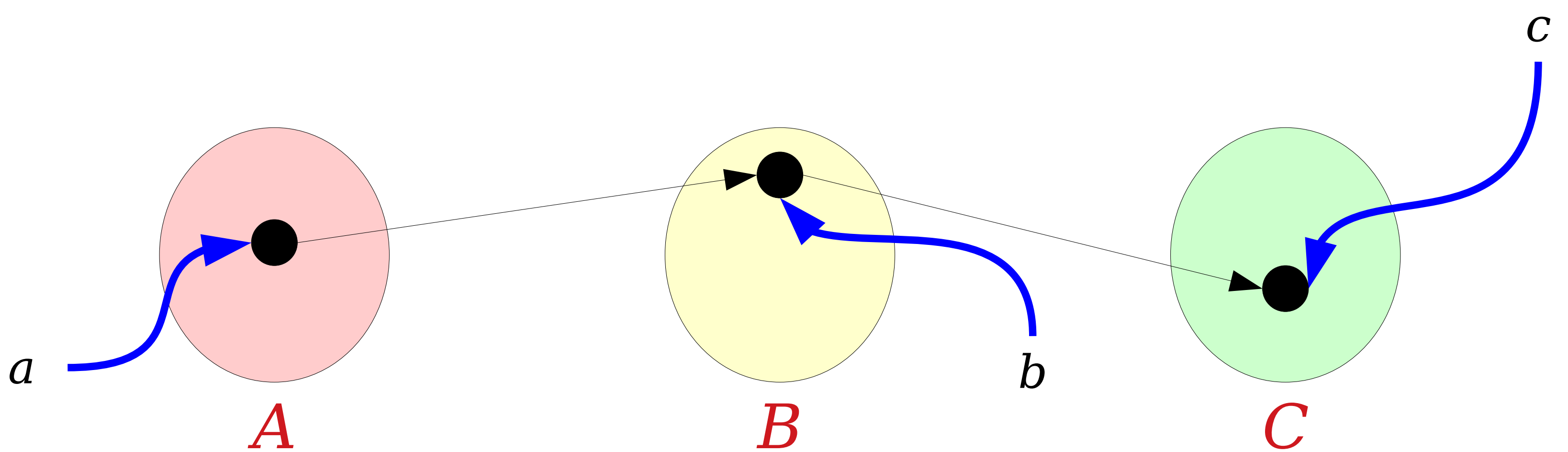
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$.



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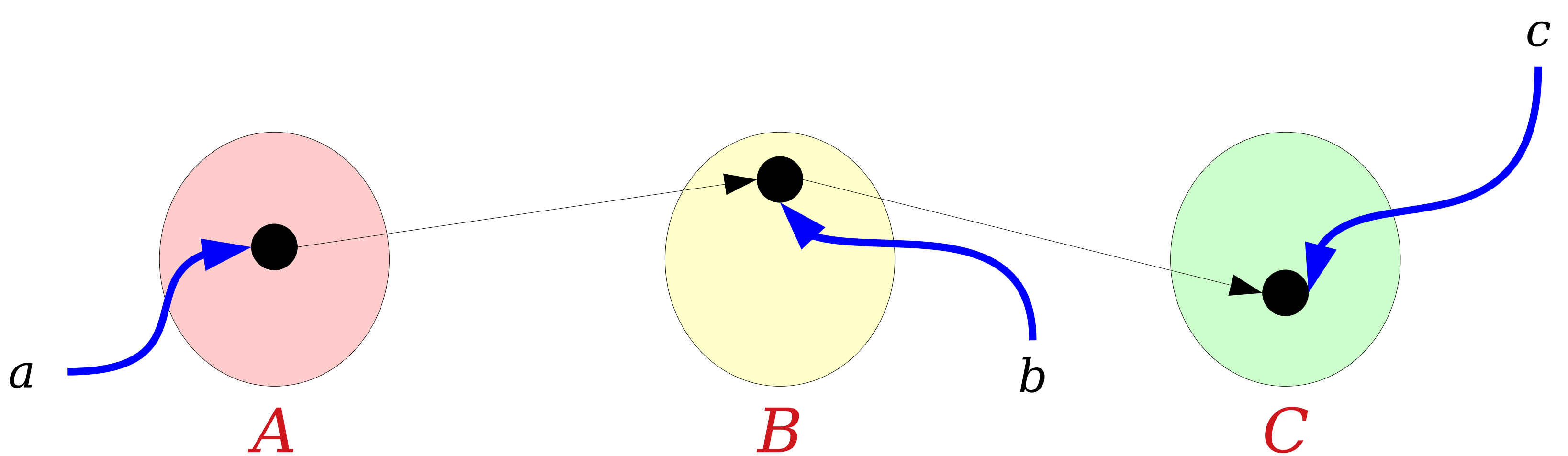
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$.



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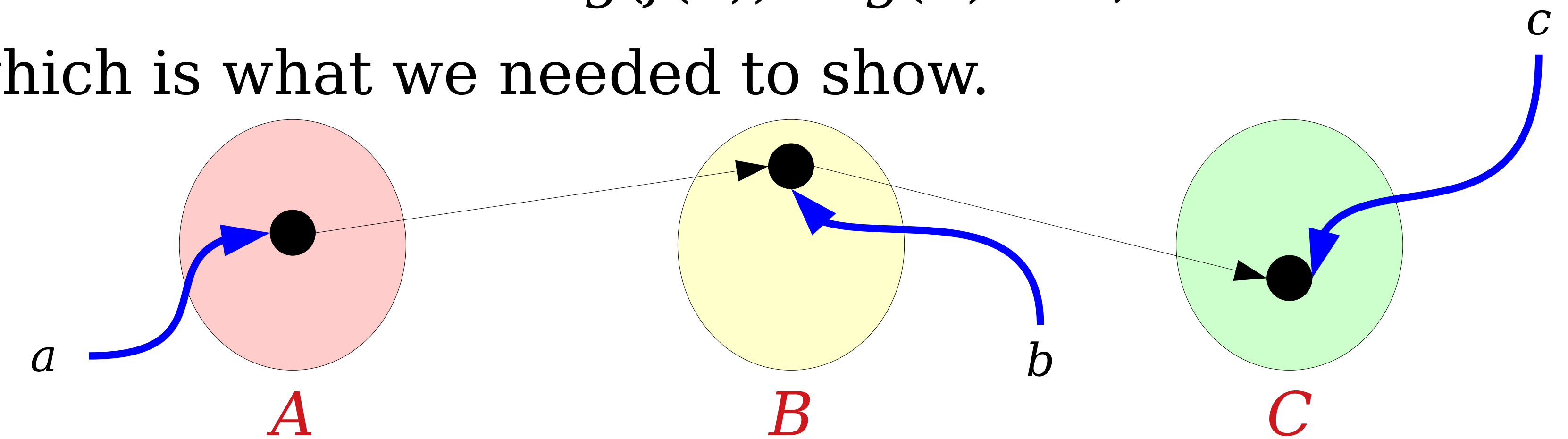
Consider any $c \in C$. Since $g : B \to C$ is surjective, there is some $b \in B$ such that $g(b) = c$. Similarly, since $f : A \to B$ is surjective, there is some $a \in A$ such that $f(a) = b$. Then we see that

$$g(f(a)) = g(b) = c,$$

which is what we needed to show.



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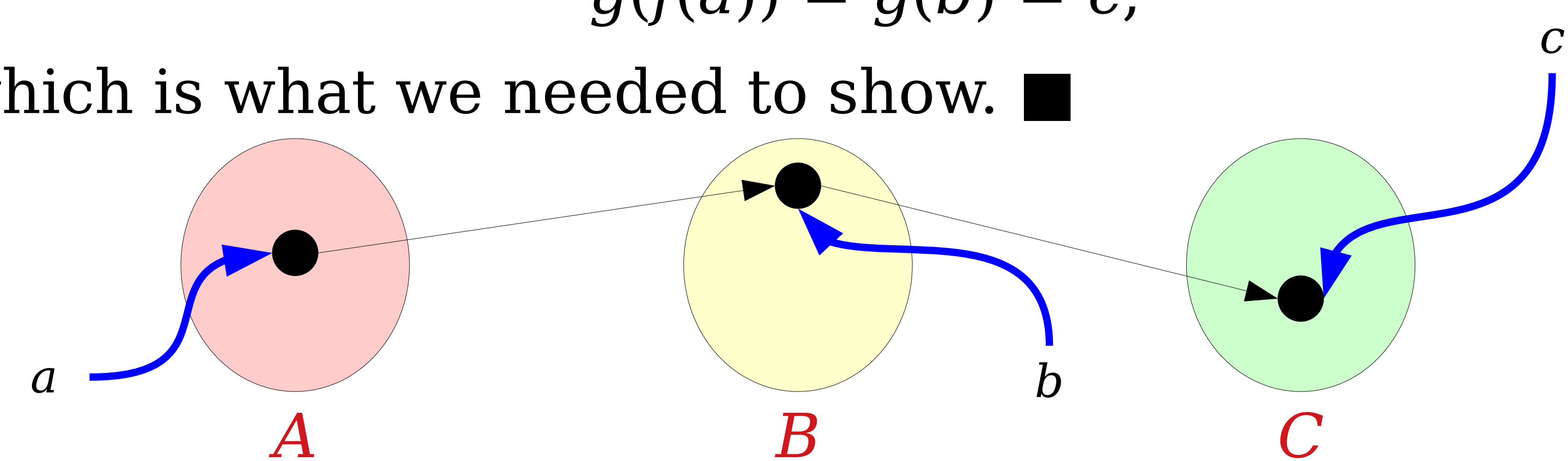

Major Ideas From Today

- Statements behave differently based on whether you're ***assuming*** or ***proving*** them.
- When you ***assume*** a universally-quantified statement, initially, do nothing. Instead, keep an eye out for a place to apply the statement more specifically.
- When you ***prove*** a universally-quantified statement, pick an arbitrary value and try to prove it has the needed property.
- As always: try concrete examples, draw pictures, etc. before you dive into writing a proof.

First-Order Logic Translation Workshop

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions
- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

write a statement in first-order logic that says “for any $n \in \mathbb{N}$, n is even if and only if n^2 is even.”

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

for any $n \in N$, n is even if and only if n^2 is even.

Using the predicate

- *Natural*(x), which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$\forall n. \ (Natural(n)$, *n is even if and only if n*^{2} *is even.*$)$

What connective goes here?

Respond at pollev.com/zhenglian740

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$\forall n.$ ($Natural(n) \to$ *n is even if and only if n^2 is even.*)

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$$\forall n.\ (Natural(n) \to (\text{n is even} \leftrightarrow n^2 \text{ is even.}))$$

Using the predicate

- *Natural*(x), which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$$\forall n.\ (Natural(n) \to (n \text{ is even} \leftrightarrow n^2 \text{ is even.}))$$

How do you express "n is even" using the given predicate and functions? *Reminder:* numbers aren't a part of first-order logic, so you can't use the number 2 in this problem.

Respond at pollev.com/zhenglian740

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$\forall n.\ (Natural(n) \to$

$(n \text{ is even} \leftrightarrow n^2 \text{ is even.})$

$)$

Using the predicate

- *Natural*(x), which states that x is a natural number

and the functions
- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$$\forall n.\ (Natural(n) \to$$
$$((\exists k.\ Natural(k) \land n = 2k) \leftrightarrow n^2 \text{ is even.})$$
$$)$$

Using the predicate

- *Natural*(x), which states that x is a natural number

and the functions

- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$$\forall n.\ (Natural(n) \to$$
$$((\exists k.\ Natural(k) \land n = k + k) \leftrightarrow n^2 \text{ is even.})$$
$$)$$

Now, complete the rest of the translation!

Respond at pollev.com/zhenglian740

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions
- $x + y$, which represents the sum of x and y, and
- $x \cdot y$, which represents the product of x and y

$$
\begin{aligned}
&\forall n.\ (Natural(n) \rightarrow \\
&\quad ((\exists k.\ Natural(k) \land n = k + k) \leftrightarrow \\
&\qquad (\exists k.\ Natural(k) \land n^2 = 2k) \\
&)
\end{aligned}
$$

Using the predicate

- $Natural(x)$, which states that x is a natural number

and the functions

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$$
\begin{aligned}
&\forall n.\ (Natural(n) \to \\
&\quad ((\exists k.\ Natural(k) \land n = k + k) \leftrightarrow \\
&\qquad (\exists k.\ Natural(k) \land n \cdot n = k + k) \\
&)
\end{aligned}
$$

Next Time

- ***Graphs***
  - A ubiquitous, expressive, and flexible abstraction!
- ***Properties of Graphs***
  - Building high-level structures out of lower-level ones!